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Critical Points of SRB Entropy

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Abstract：We provide a different proof for the following result in [5]: if a smooth expandingcircleendomorphismisacriticalpointoftheSRBentropyfunctional,thenitmustbesmoothlyconjugatetothelinearendomorphism.

1Introduction

Amongst all the invariant measures for differentiable dynamical systems, the Sinai-Ruelle-Bowen (SRB) measure is a key tool to understand the natural laws behind physical models. There has been growing interests on the study of SRB measures under perturbations, in terms of properties such as stochastic stability, linear response, Lyapunov exponents and entropy.

We are particularly interested in the Kolmogorov-Sinai entropy of the SRB measures, which defines a nonlinear functional in some topological class of dy- namical systems. Motivated by the Gallavotti-Cohen Chaotic Hypothesis, a conjecture was proposed: In typical class of chaotic systems, the SRB entropy functional does not have nontrivial local maximum. In other words, typically a local maximum of the SRB entropy functional must be a global maximum.

Some positive results have been achieved in low dimensional systems, such as the class of smooth circle expanding endomorphisms by Jiang [5] and Markov transformations on a closed interval by Jiang and Lopez [6]). There is also a recent affirmative result by Saghin, Valenzuela-Henriquez, and Vasquez [10] in the category of C^3family of transitive Anosov maps on two torus.

In this note, we shall restrict ourselves in the class of smooth expanding circle endomorphisms that had been considered by Jiang in [5], in which he prove the

conjecture using the Lagrange multiplier theorem in Banach spaces to obtain necessary conditions for the critical values of the entropy. We shall provide here a quite different proof by using a special perturbation, together with the analytic formula of linear response. Our proof involves with an explicit construction for the perturbation, which may be generalized to higher dimensional cases in the future work.

The paper is organized as follows: firstly, we introduce all necessary notations and state our main results in Section 2; next, we explain the special perturbation and the linear response formula in Section 3; finally, we present the proof of our main theorem in Section 4.

2Notations and MainResults

In this note, we consider the space of smooth expanding endomorphisms on the unit circle, and study the critical points of the SRB entropy functional in this space. We first introduce the following notations.

(1)Let T be the unit circle. The universal covering space of T is the real line R, with the natural projection $\Pi: R\to T$ given by $\Pi(x) = \exp(i2\pi x)$. In this way, we may regard T as the unit interval [0, 1] with 0 ∼ 1.

(2)Let $k \geqslant 2$ be a fixed integer, and let F be the space of C3 endomorphisms of T with degree k. Since any endomorphism $f \in F$ can be lifted to a function on the covering space R, we shall identify f with its lifting. That is, by abusing notations if there are no confusions, we may write

$$\mathcal{F} = \left\{f \in C^3(\mathbb{R}): \quad f(x+1) = f(x) + k \quad \text{for any } x \in \mathbb{R}\right\}.$$

(3)TomakeaperturbationinthespaceF,weintroducethespace

$$\mathcal{G} := \left\{g \in C^3(\mathbb{R}): \quad g(x+1) = g(x) \quad \text{for any } x \in \mathbb{R}\right\},$$

which is the space of C^3 period one function on R. It is easy to verify that $F + G \subseteq F$ and $G \circ F \subseteq G$, that is, $f + g \in F$ and $g \circ f \in G$ for any $f \in F$ and $g \in G$.

(4)Let E be the subspace of F such that every $f \in E$ is uniformly expanding, i.e., $\min_{x\in R} f'(x) > 1$. It is clear that E is an open, path connected subset of F, while an endomorphism in ∂E need not be uniformly expanding.

Next we recall some definitions and results from smooth ergodic theory, for which the readers may consult the references [1, 3, 9, 12].

For any $f \in F$, if an f-invariant probability measure μ is absolutely con-tinuous with respect to the Lebesgue measure on T, then it is called a Sinai- Ruelle-Bowen (SRB) measure of f. It is well known that every expanding circle endomorphism $f \in E$ admits a unique SRB measure μf on T, such that the density function $\rho f(x) = d\mu f(x)/dx$ is C^2 smooth.

Given any $f \in E$, we denote $H(f) := h\mu f(f)$ the Kolmogorov-Sinai entropy of the SRB measure μf, which we shall call the SRB entropy of f. The following properties of the SRB entropy functional $H: E \to R$ are well known:

(1)H(•) is analytic on E (see e.g. [8]);

(2)the range of H(•) is (0, log k]. Indeed, the maximum is due to the variational principle and the fact that the topological entropy is log k, and the infimum being zero were established in [4, 7];

(3)by the Roklin–Pesin entropy formula, we have

$$H(f) = \int_{\mathbb{T}} \log |f'(x)| d\mu_f(x) = \int_{\mathbb{T}} \rho_f(x) \log(f'(x)) dx. \quad (2.1)$$

We say that $f \in E$ is a *critical point* of the SRB entropy functional $H(\cdot)$, if for any C^1 path $\{f_t\}_{t\in(-\epsilon,\epsilon)}$ with $f_0 = f$ in the space E, where $\epsilon > 0$.

$$\left.\frac{d}{dt}\right|_{t=0} H(f_t) = 0,$$

Note that any $f \in E$ is topologically conjugate to the linear circle endomor– phism $E_k(x) = kx$ (mod 1). Our main theorem establishes the smooth rigidity related to the SRB entropy functional, which is stated as follows.

Theorem 2.1. If f $\in$ E is a critical point of the entropy function H($\cdot$), then f is C^3 conjugate to the linear endomorphism E_k.

The following corollary is immediate.

Corollary 2.2. *If the SRB entropy of f $\in$ E equals to* log *k, then f is C^3*

conjugate to the linear endomorphism E_k.

3SomePreliminaries

3.1A SpecialPerturbation

It was pointed out by Shub–Sullivan [11] that any $f \in E$ is C^3 conjugate to an $f \in E$ which preserves the Lebesgue measure (see also the Moser' s trick explained in [4 6]). More precisely, we recall that ρ_f is the density of the

unique SRB measure of f, which can be viewed as a C^2 period one function on R. We define $f = h \circ f \circ h^{-1}$, where h is given by

$$h(x) = \int_0^x \rho_f(z) dz, \quad \text{for any } x \in \mathbb{R}.$$

It is not hard to verify that h is a C^3 diffeomorphism on T with degree one, i.e.,

$h(x + 1) = h(x) + 1$ for any $x \in$ R, and thus f also belongs to E .

For any $f \in E$ and any $g \in G$, we first make an additive perturbation on f by g, that is,for some sufficiently small $\epsilon > 0$. We then apply a smooth conjugation on f_t by h, that is, $f_t = h^{-1} \circ f_t \circ h$. It is clear that both f_t and f_t are C^1 paths in E .

$$\tilde{f}_t = \tilde{f} + tg \circ \tilde{f}, \quad \text{where } |t| < \epsilon, \quad (3.1)$$

Let μ_{f_t} and $\mu_{\tilde{f}_t}$ be the SRB measures of f_t and f_t respectively. We have that $h_*(\mu_{f_t}) = \mu_{\tilde{f}_t}$ due to the uniqueness of SRB measures for both f_t and f_t, as well as the fact that h is absolutely continuous with respect to the Lebesgue measure on T. Therefore, by (2.1) we obtain that

$$\begin{aligned} H(f_t) = H\left(\tilde{f}_t\right) &= \int_{\mathbb{T}} \rho_t(x) \log \left(\tilde{f}'(x) + tg'\left(\tilde{f}(x)\right)\tilde{f}'(x)\right) dx \\ &= \int_{\mathbb{T}} \rho_t(x) \left[\log \tilde{f}'(x) + \log\left(1 + tg'\left(\tilde{f}(x)\right)\right)\right], \end{aligned} \quad (3.2)$$

where we denote ρ_t the density of the SRB measure of $\tilde{f}_t$. Note that $\rho_0 = \rho_{\tilde{f}} \equiv 1$.

3.2Linear ResponseFormula

To compute the derivative of the SRB entropy functional along a particular path, we shall apply the linear response formula.

More precisely, let f_t be the path given by (3.1), and let ρ_t be the SRB density of f_t. The mapping $t \mapsto \rho_t$ is differentiable at $t = 0$, and the derivative at $t = 0$ is given by

$$\xi := \partial_t \rho_t|_{t=0} = -\sum_{n=0}^{\infty} \mathcal{L}^n (g\rho_0)' = -\sum_{n=0}^{\infty} \mathcal{L}^n g', \quad (3.3)$$

where L is transfer operator associated with $\tilde{f}$. Here L is defined by

$$\mathcal{L}\varphi(x) = \sum_{y\in\tilde{f}^{-1}(x)} \frac{\varphi(y)}{\tilde{f}'(y)}, \quad \text{for any } \varphi \in C(\mathbb{T}).$$

Equivalently, the transfer operator L can be characterized by the following du– ality relation:

$$\int_{\mathbb{T}} \mathcal{L}\varphi(x)\psi(x)dx = \int_{\mathbb{T}} \varphi(x)\psi \circ \tilde{f}(x)dx, \quad \text{for any } \varphi, \psi \in C(\mathbb{T}). \quad (3.4)$$

For more details on transfer operators, see the references [1, 12].

The linear response formula given by (3.3) is well known, but is usually presented in an integral form (see e.g. the expository survey [2]). It is worth

pointing out that the series in (3.3) converges absolutely since the transfer op– erator L is uniformly contracting on the space

$$\mathcal{G}_0 := \left\{\varphi \in \mathcal{G} : \int \varphi(x)dx = 0\right\},$$

and it is clear that $g' \in G_0$ for any $g \in G$.

4Proof of Theorem2.1

Suppose now $f \in E$ is a critical point of the SRB entropy functional. For any $g \in G$, we let f_t be the C^1 path constructed in (3.1). A direct calculation by taking derivative of (3.2) yields that

$$\begin{aligned} 0 = \left.\frac{d}{dt}\right|_{t=0} H(f_t) &= \int_{\mathbb{T}} \xi(x) \log \tilde{f}'(x)dx + \int_{\mathbb{T}} \rho_0(x) g'(\tilde{f}(x))dx, \\ &= \int_{\mathbb{T}} \xi(x) \log \tilde{f}'(x)dx. \end{aligned} \quad (4.1)$$

where ξ is given by (3.3). Note that the last term vanishes because $\rho_0 \equiv 1$ and f preserves the Lebesgue measure so that

$$\int_{\mathbb{T}} g'(\tilde{f}(x))dx = \int_{\mathbb{T}} g'(x)dx = g(1) - g(0) = 0.$$

We now choose a particular $g \in G$ as follows. We define

$$\psi(x) := \log \tilde{f}'(x) - \int_{\mathbb{T}} \log \tilde{f}'(z)dz, \quad \text{for any } x \in \mathbb{R}. \quad (4.2)$$

Since $\int_{\mathbb{T}} \psi(x)dx = 0$, it is easy to see that the map

$$g(x) := \int_0^x \psi(z)dz, \quad \text{for any } x \in \mathbb{R},$$

belongs to G. Applying the duality relation in (3.4), we obtain from (4.1) that

$$\begin{aligned} 0 = -\int_{\mathbb{T}} \sum_{n=0}^{\infty} (\mathcal{L}^n g')(x) \cdot \log \tilde{f}'(x)\, dx &= -\sum_{n=0}^{\infty} \int_{\mathbb{T}} g'(x) \cdot \log \tilde{f}' \circ \tilde{f}^n(x)\, dx \\ &= -\sum_{n=0}^{\infty} \int_{\mathbb{T}} \psi(x) \cdot \psi \circ \tilde{f}^n(x) dx \\ &= -\frac{1}{2}\left(\int_{\mathbb{T}} \psi^2(x)dx + \sigma_\psi^2\right). \end{aligned}$$

Here σ^2_ψ stands for the variance given by the Green-Kubo formula:

$$\begin{aligned} \sigma_\psi^2 &:= \int_{\mathbb{T}} \psi^2(x)dx + 2\sum_{n=1}^{\infty} \int_{\mathbb{T}} \psi(x) \cdot \psi \circ \tilde{f}^n(x)dx \\ &= \lim_{n\to\infty} \frac{1}{n} \int \left(\sum_{k=0}^{n-1} \psi \circ f^k(x)\right)^2 dx, \end{aligned}$$

where is always non-negative. Therefore, we must have implies that $\psi \equiv 0$.

$\int_{\mathbb{T}} \psi 2(x)dx = 0$, which

By the definition of ψ in (4.2), we conclude that $\log f'$ and thus f' must be constant. On the other hand, since $f \in E$ is of degree k, we have

$$\int_{\mathbb{T}} \tilde{f}'(x)dx = \tilde{f}(1) - \tilde{f}(0) = k.$$

Then $f'(x) \equiv k$ and thus $f(x) = kx + \alpha$ for some $\alpha \in \mathbb{R}$. Define a circle rotation by $\beta(x) := x + \alpha/(k - 1) \pmod 1$, then we get $\beta \circ f \circ \beta^{-1} = E_k$, that is, f is C^3 conjugate to E_k. Hence $f = h \circ f \circ h^{-1}$ is also C^3 conjugate to E_k.

The proof of Theorem 2.1 is complete.

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