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全排列的和的均值问题

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摘　要：本文研究了2017年4月7日举行的欧几里得数学竞赛第9题，将全排列的和的均值问题从有限个自然数推广到任意n个数，其中数字不仅限于自然数，还可以是普通的数列，例如等差数列，且结论的形式更加简洁。
关键词：欧几里得数学竞赛；全排列的和；均值

The Average Value of a Sum over all Permutations

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Abstract：We study the 9th question of the Euclid Contest in April 7，2017 of University of Waterloo. We can extend the method of determining the average value of a sum over all permutations from certain numbers to n numbers. Meanwhile，the numbers are not only natural numbers，but also can be a common series，such as an arithmetic sequence. The form of the average value is concise.
Keywords：Euclid Contest；Sum over all Permutations；The Average Value

1. The 9th question of the Euclid Contest in April 7，2017 of University of Waterloo

Apermutation of a list of numbers is an ordered arrangement of the numbers in that list. For example，3，2，4，1，6，5 is a permutation of 1，2，3，4，5，6. We can write this permutation as a_1，a_2，a_3，a_4，a_5，a_6， where $a_1=3$，$a_2=2$，$a_3=4$，$a_4=1$，$a_5=6$， and $a_6=5$.

（a） Determine the average value of $|a_1-a_2|+|a_3-a_4|$ over over all permutations a_1，a_2，a_3，a_4 of 1，2，3，4.

（b） Determine the average value of $a_1-a_2+a_3-a_4+a_5-a_6+a_7$ over all permutations a_1，a_2，a_3，a_4，a_5，a_6，a_7 of 1，2，3，4，5，6，7.

（c） Determine the average value of $|a_1-a_2|+|a_3-a_4|+\cdots+|a_{197}-a_{198}|+|a_{199}-a_{200}|$ over all permutations a_1，a_2，a_3，$\cdots$，a_{199}，a_{200} of 1，2，3，4，⋯，199，200.

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2. Theorem

Theorem 1 The average value of $a_1-a_2+a_3-a_4+\cdots+(-1)^{n+1}a_n$ over all permutations a_1，a_2，$\cdots$，a_n of 1，2，⋯，n is $\frac{n+1}{2}$ when n is even and 0 when n is odd.

Proof：

There aren！permutations of 1，2，⋯，n. We determine the average value of $a_1-a_2+a_3-a_4+\cdots+(-1)^{n+1}a_n$ over all of these permutations by determining the sum of all n！values of this expression and dividing by n！.

To determine the sum of alln！values，we write them in a matrix as follow.

$$\begin{pmatrix} a_1 & -a_2 & a_3 & -a_4 & \cdots & (-1)^{n+1}a_n \\ a_1 & -a_3 & a_2 & -a_4 & \cdots & (-1)^{n+1}a_n \\ a_1 & -a_3 & a_4 & -a_2 & \cdots & (-1)^{n+1}a_n \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_n & -a_{n-1} & a_{n-2} & -a_{n-3} & \cdots & (-1)^{n+1}a_1 \end{pmatrix}$$

We determine the sum of the values of the first column and call this totals，then the sum of the second column will be −s，and so on. The sum of the values of the last column is $(-1)^{n+1}s$. Obviously，the average value of the sum of all n！permutations is $\frac{s}{n!}$ when n is even and 0 when n is odd. So we need to determine the value of s.

There are (n−1)！a_1 in the first column，(n−1)！a_2 in the first column，and so on. Thus，

$$s=(a_1+a_2+\cdots+a_n)\cdot(n-1)!=(1+2+\cdots+n)\cdot(n-1)!=\frac{n(n+1)}{2}\cdot(n-1)!=\frac{(n+1)!}{2}$$

Therefore，the average value of the expression is $\frac{n+1}{2}$ when n is even and 0 when n is odd.

Theorem 2 The average value of $|a_1-a_2|+|a_3-a_4|+\cdots+|a_{n-1}-a_n|$ over all permutations a_1，a_2，$\cdots$，a_n of 1，2，

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$\cdots$, n is $\frac{n(n+1)}{6}$. (n is even).

Proof:

There aren! permutations of 1, 2, $\cdots$, n. According to the proof of Theorem 1, we can get a matrix with n! rows and $\frac{n}{2}$ columns. We determine the sum of the values of the first column and call this total s, then the sun of other columns will also be s. Thus, the average value of the sum of all n! permutations is $\frac{\frac{n}{2}\cdot s}{n!}$.

There are (n−2)! permutations with $a_1=i$, $a_2=j$. Since $|i-j|=|j-i|$, then there are 2 [(n−2)!] permutations with $|a_1-a_2|=|i-j|$ that come from a_1 and a_2 equalling i and j in some order. Therefore, we may assume that i > j and note that s equals 2 [(n−2)!] times the sum s_1 of i−j over all possible pairs i > j. Thus, the average value of the sum of all n! permutations is $\frac{\frac{n}{2}\cdot s}{n!}=\frac{\frac{n}{2}\cdot 2\cdot(n-2)!\cdot s_1}{n!}=\frac{s_1}{n-1}$. So we need to determine the value of s_1.

$$s_1=(a_2-a_1)+(a_3-a_1)+\cdots+(a_n-a_1)+(a_3-a_2)+(a_4-a_2)+\cdots+(a_n-a_2)+\cdots+(a_n-a_{n-1})$$
$$=(2-1)+(3-1)+\cdots(n-1)+(3-2)+(4-2)+\cdots+(n-2)+\cdots+[n-(n-1)]$$
$$=1\cdot(n-1)+2\cdot(n-2)+\cdots+(n-1)\cdot 1$$
$$=[n-(n-1)](n-1)+[n-(n-2)](n-2)+\cdots+(n-1)\cdot 1$$
$$=n[(n-1)+(n-2)+\cdots+1]-[(^{n}-1)2+(^{n}-2)2+\cdots+1^2]$$
$$=n\cdot\frac{(n-1)n}{2}-\frac{(n-1)n(2n-1)}{6}$$
$$=\frac{n(n-1)(n+1)}{6}$$

Therefore, the average value of the expression is

$$\frac{s_1}{n-1}=\frac{1}{n-1}\cdot\frac{n(n-1)(n+1)}{6}=\frac{n(n+1)}{6}.$$

3. Conclusion

Conclusion 1 The average value of $a_1-a_2+a_3-a_4+\cdots+(-1)^{n+1}a_n$ over all permutations a_1, a_2, $\cdots$, a_n of the first n of the sequence $\{b_n\}$ is $\frac{T_n}{n}$ when n is even and 0 when n is odd. T_n is the sum of the first n terms of $\{b_n\}$.

Conclusion 2 The average value of $|a_1-a_2|+|a_3-a_4|+\cdots+|a_{n-1}-a_n|$ over all permutations a_1, a_2, $\cdots$, a_n of the first n of the arithmetic sequence $\{b_n\}$ is $\frac{n(n+1)}{6}\cdot d$. d is the common difference of $\{b_n\}$.